Variations on the Erdős Distinct-Sums Problem

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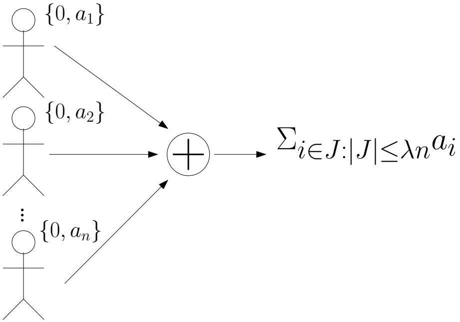
1. Information-theoretic Interpretation
2. Formulation of the original problem by Erdős
3. Some known results from literature
4. Variations on the original problem
5. Our results

Signaling over a multiple access channel. Let $\{a_1, ..., a_n\}$ be a set, where $a_i \in \mathbb{Z}^k$ for some $k \geq 1$, such that the sums of up to λn elements of the set be distinct with $0 < \lambda < 1$. We have n trasmitters.



Each one can send a signal of amplitude a_i to the base station saying that it wants to start a communication session.

Formulation of the original problem

Let $\{a_1, ..., a_n\}$ be a set of positive integers with $a_1 < \ldots < a_n$ such that all 2^n subset sums are distinct.

Conjecture.

A famous conjecture by Erdős states that $a_n > c \cdot 2^n$ for some constant c.

Theorem.

The best results known to date are of the form $a_n > c \cdot 2^n/\sqrt{n}$ for some constant c.

Improving the factor $\sqrt{n}$ is a very hard task and so only the constant c has been improved in the past 65 years.

Known bounds from literature

Lower bounds on a_n

- Trivial one $\to a_n \geq \frac{1}{n}2^n$
- Erdős and Moser (1955) $\to a_n \geq \frac{1}{4} \cdot \frac{2^n}{\sqrt{n}}$
- Alon and Spencer (1981) $\to a_n \geq (1+o(1))\frac{2}{3\sqrt{3}} \cdot \frac{2^n}{\sqrt{n}}$
- Elkies (1986) $\to a_n \geq (1+o(1))\frac{1}{\sqrt{2\pi}} \cdot \frac{2^n}{\sqrt{n}}$
- Guy (1981) $\to a_n \geq (1+o(1))\frac{1}{\sqrt{3}} \cdot \frac{2^n}{\sqrt{n}}$
- Dubroff, Fox and Xu (2020) $\to a_n \geq (1+o(1))\sqrt{\frac{2}{\pi}} \cdot \frac{2^n}{\sqrt{n}}$

Upper bounds on a_n

- Trivial one $\to a_n \leq 2^{n-1}$ (take each $a_i = 2^{i-1}$)
- Bohman (1998) $\to a_n \leq 0.22002 \cdot 2^n$

Variations on the original problem

First variation.
The distinct-sums condition is weakened by only requiring that the sums of up to λn elements of the set be distinct with $0 < \lambda < 1$.

Second variation.
The elements $a_i \in \mathbb{Z}^k$ for some $k \geq 1$.

More formally...

Problem.

Let $\mathcal{F}_{\lambda,n}$ be the family of all subsets of $\{1, \ldots, n\}$ whose size is smaller than or equal to λn. We are interested in the minimum M such that there exists a sequence $\Sigma = (a_1, \ldots, a_n)$ in $\mathbb{Z}^k$, $a_i \in [0, M]^k$ $\forall i$, such that for all distinct $A_1, A_2 \in \mathcal{F}_{\lambda,n}$, $S(A_1) \neq S(A_2)$, where

$$S(A) = \sum_{i \in A} a_i \,.$$

Remark.

For $\lambda = 1$ and $k = 1$ we obtain the same problem as the original one.

Trivial Lower Bounds on M

Proposition.

Let $\Sigma = (a_1, \ldots, a_n)$ be an $\mathcal{F}_{\lambda,n}$-sum distinct sequence in $\mathbb{Z}^k$ that is M-bounded. Then

$$M \geq (1 + o(1)) \cdot \begin{cases} \frac{1}{\lceil \lambda n \rceil \sqrt[k]{2\pi n \lambda(1-\lambda)}} 2^{nh(\lambda)/k} & \text{if } \lambda < 1/2; \\ \frac{1}{\lceil \lambda n \rceil} \cdot 2^{(n-1)/k} & \text{if } 1/2 \leq \lambda < 1; \\ \frac{1}{n} \cdot 2^{n/k} & \text{if } \lambda = 1; \end{cases}$$

Proof.

Since the maximum possible sum is at most $\lceil \lambda n \rceil M$ in each component, by the pigeonhole principle, we have that

$$M^k \geq \frac{1}{\lceil \lambda n \rceil^k} \sum_{i=0}^{\lceil \lambda n \rceil} \binom{n}{i}.$$

□

Following the idea of Dubroff, Fox and Xu the previous bounds for $k = 1$ and $\lambda \geq 1/2$ can be improved as follow

Theorem

Let $\Sigma = (a_1, \ldots, a_n)$ be an $\mathcal{F}_{\lambda,n}$-sum distinct sequence in $\mathbb{Z}$ that is M-bounded. Then

$$M \geq (1 + o(1)) \cdot \begin{cases} \frac{1}{\sqrt{2\pi n}} \cdot 2^n & \text{if } \lambda = 1/2; \\ \sqrt{\frac{2}{\pi n}} \cdot 2^n & \text{if } \lambda \in]1/2, 1]. \end{cases}$$

Lower bound - Variance method for $k > 1$

Using the variance method, it is possible, to improve the previous bounds for $k > 1$ and $\lambda \geq 1/2$.

Theorem

Let $\lambda \geq 1/2$ and let $\Sigma = (a_1, \ldots, a_n)$ be an $\mathcal{F}_{\lambda,n}$-sum distinct sequence in $\mathbb{Z}^k$ that is M-bounded. Then

$$M \geq (1+o(1)) \cdot \begin{cases} \sqrt{\frac{4}{\pi n(k+2)}} \cdot \Gamma(k/2+1)^{1/k} \cdot 2^{n/k} & \text{if } \lambda = 1; \\ \sqrt{\frac{4}{\pi n(k+2)}} \cdot \Gamma(k/2+1)^{1/k} \cdot 2^{(n-1)/k} & \text{if } 1/2 \leq \lambda < 1; \end{cases}$$

where Γ is the gamma function.

Variance method for $k > 1$ and $1/2 \leq \lambda \leq 1$

Sketch of the proof.

Consider a random variable $X = \sum_{i=1}^{n} \epsilon_i a_i$ where the random vectors $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ are uniformly distributed over the set $\mathcal{F}_{\lambda,n}$. It can be proved that $\mathbb{E}[\epsilon_i \epsilon_j] \leq 0$ for each $i \neq j$. Hence $\sigma^2 \leq 1/4 \sum_{i=1}^{n} |a_i|^2 \leq 1/4nkM^2$.

The variance can be lower bounded by placing each sum within a ball of the smallest possible radius R centered at $\mu := \mathbb{E}[X]$.

$$\begin{aligned} \sigma^2 &\geq \frac{(1 + o(1))}{|\mathcal{F}_{\lambda,n}|} \int_0^R S_{k-1}(\rho)\rho^2 d\rho \\ &\geq \frac{(1 + o(1))}{|\mathcal{F}_{\lambda,n}|} \frac{k\pi^{k/2}}{\Gamma(k/2 + 1)} \frac{R^{k+2}}{k + 2}. \end{aligned}$$

Using the polynomial method (Alon's combinatorial nullstellensatz) we get the following theorem.

Theorem

For any $\lambda < 1/3$*, there exists a sequence* $\Sigma = (a_1, \ldots, a_n)$ *that is* M*-bounded positive integers and* $\mathcal{F}_{\lambda,n}$*-sum distinct with*

$$M \geq \lambda^3 n^2 2^{f(\lambda)n},$$

where $f(\lambda) = -2\lambda \log_2 \lambda - (1-2\lambda)\log_2(1-2\lambda)$.

Remark.

The previous bound is non-trivial for $\lambda < 3/25$.

Upper bound - Probabilistic method

Using the probabilistic method we get an improvement on the trivial bound (i.e., $c \cdot 2^{n/k}$) for $k > 1$ and $\lambda < 3/25$.

Theorem

Let

$$C_{\lambda,n} = \sqrt[k]{\frac{\lambda^2 n^2}{2\tau_\lambda} 2^{f(\lambda)\tau_\lambda}} \text{ and } \tau_\lambda = \left\lceil \frac{1}{\log_e 2 \cdot f(\lambda)} \right\rceil,$$

where $f(\lambda) = -2\lambda \log_2 \lambda - (1-2\lambda)\log_2(1-2\lambda)$.

Then there exists a sequence $\Sigma = (a_1, \ldots, a_n)$ *of* $\left(C_{\lambda,n} \cdot 2^{f(\lambda)n/k}\right)$*-bounded elements of* $\mathbb{Z}^k$ *that is* $\mathcal{F}_{\lambda,n}$*-sum distinct.*

Improvements for $k = 1$ and $\lambda \in]3/25, 1/4]$

Using the Bohman construction it can be shown that if $\lambda < 1/4$, if n is big enough, there exists a sequence $\Sigma = (a_1, \dots, a_n)$ of M-bounded integers that is $\mathcal{F}_{\lambda,n}$-sum distinct with

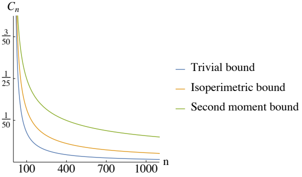
$$M = \frac{0,22096}{2} \cdot 2^n \,,$$

while for $\lambda < 1/8$ we get

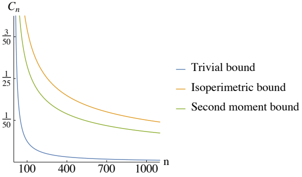
$$M = \frac{0,22096}{4} \cdot 2^n \,.$$

Remark.
For $\lambda < 1/4$ we can insert an additional a_i to the sequence found by Bohman while for $\lambda < 1/8$ two elements can be added without violating the sum-distinct property.

Figure: Sub-exponential factor of the lower bounds for $1/2 \leq \lambda \leq 1$.

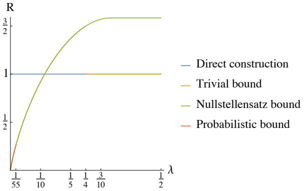


(a) $k = 1$

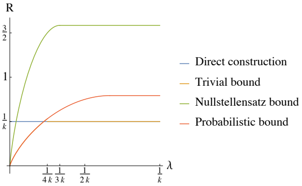


(b) $k > 1$

Figure: Exponent of the upper bounds for $k = 1$ and for $k > 1$.



(a) $k = 1$



(b) $k > 1$