On Sequences in Cyclic Groups with Distinct Partial Sums

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Combinatorics, Mantova, 2022

Some notation

Let S be a subset of size k of a finite abelian group G of order n.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ be an ordering of the elements of S and define its **partial sums** by $\mathbf{y} = (y_0, y_1, y_2, \ldots, y_k)$ where

$$y_0 = 0, y_1 = x_1, y_2 = x_1 + x_2, \ldots, y_k = x_1 + x_2 + \ldots + x_k$$

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Call an ordering of S with distinct partial sums, with the only exception that $\sum S$ can be equal to 0, a **sequencing**.

If S admits a sequencing then it is called **sequenceable**.

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Conjecture

Let G be a finite abelian group. Then, every subset $S \subseteq G \setminus \{0\}$ is sequenceable.

Some motivation

This problem has connections to:

- Heffter arrays
- Non-zero sum Heffter arrays
- graph-decomposition
- graceful-labelings
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For example:

The truth of the conjecture would show that any $\mathrm{NH}(n;k)$ provides two orthogonal path decompositions of the complete graph K_{2nk+1}.

Some known results

Let G be an abelian group of order n and $S \subseteq G \setminus \{0\}$ of size k. Then S is sequenceable in the following cases:

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7. $n \leq 25$ when G is cyclic and $\sum S = 0$ [Archdeacon, Dinitz]

Theorem (Alon '99)

Let $\mathbb{F}$ be a finite field, $f(x_1, x_2, \ldots, x_k)$ be a polynomial in $\mathbb{F}[x_1, x_2, \ldots, x_k]$ and let $C_1, C_2, \ldots, C_k$ be subsets of $\mathbb{F}$. If there exists a monomial of maximum degree with non-zero coefficient in f that divides $x_1^{|C_1|-1} \cdots x_k^{|C_k|-1}$ then there are $e_1 \in C_1, \ldots, e_k \in C_k$ such that $f(e_1, e_2, \ldots, e_k) \neq 0$.

Call $x_1^{|C_1|-1} \cdots x_k^{|C_k|-1}$ the **bounding monomial**.

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Let p be prime and take $S \subseteq \mathbb{Z}_p \setminus \{0\}$ with $|S| = k$. To use the NVC, we take each $C_i = S$ and we need a polynomial f that is nonzero exactly when $(x_1, \ldots, x_k)$ is a sequencing of S.

Hicks, Ollis and Schmitt introduced the following polynomial

$$\prod_{1\le i<j\le k}(x_j-x_i)\prod_{\substack{0\le i<j\le k\\(i,j)\neq(0,k),j\neq i+1}}(y_j-y_i)$$

That gives

$$f=\prod_{1\le i<j\le k}(x_j-x_i)\prod_{0\le i<j\le k,j\neq i+1}(x_{i+1}+\ldots+x_j)\Big/(x_1+\ldots+x_k)$$

To apply the NVC we need a nonzero coefficient on a monomial in f that divides $x_1^{k-1}\cdots x_k^{k-1}$ which has degree $k(k-1)$. Since $\deg(f)=k(k-1)-1$ there are k monomials that could work.

Theorem (Hicks, Ollis and Schmitt '18)

For $G = \mathbb{Z}_p$ with p an odd prime and $|S| \leq 10$, S is sequenceable.

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For $S \subseteq \mathbb{Z}_p \setminus \{0\}$, $|S| = 12$ we have

monomial	coefficient
$x_1^{10} x_2^{11} x_3^{11} \cdots x_{12}^{11}$	$2^4 \cdot 3 \cdot 29 \cdot 12953077208391719881$
$x_1^{11} x_2^{10} x_3^{11} \cdots x_{12}^{11}$	$2^3 \cdot 3 \cdot 277 \cdot 1901 \cdot 786640832519761$

$\mathbb{Z}_p \times \mathbb{Z}_2$ example

Suppose we have $S \subseteq \mathbb{Z}_p \times \mathbb{Z}_2 \setminus \{(0,0)\}$, $|S| = 5$ and 3 of the elements are in coset 0 and 2 of them in coset 1.

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We look for a sequencing of the form

$$\mathbf{x} = ((x_1, 0), (x_2, 1), (x_3, 0), (x_4, 0), (x_5, 1))$$

which has partial sums

$$\mathbf{y} = ((y_0, 0), (y_1, 0), (y_2, 1), (y_3, 1), (y_4, 1), (y_5, 0))$$

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Factors of the polynomial needed:

$$(x_3 - x_1)(x_4 - x_1)(x_4 - x_3)(x_5 - x_2)(y_5 - y_1)(y_4 - y_2)$$

This has degree 6. The monomial $x_1^2 x_3^2 x_4 x_5$ has coefficient -1 and it divides $x_1^2 x_2 x_3^2 x_4^2 x_5$. Therefore S has a sequencing.

The **type** of $S \subseteq \mathbb{Z}_p \times \mathbb{Z}_t \setminus \{(0,0)\}$ is $(\lambda_0, \ldots, \lambda_{t-1})$, where λ_i is the number of elements of S in coset i.

Given $k = |S|$, for each type we choose an ordering $\mathbf{a}$ of the cosets (with repetition) with partial sums $\mathbf{b}$. Following the previous example we construct the polynomial

$$f = \prod_{\substack{1 \leq i < j \leq k \\ a_i = a_j}} (x_j - x_i) \prod_{\substack{0 \leq i < j \leq k \\ b_i = b_j \\ j \neq i+1 \\ (i,j) \neq (0,k)}} (y_j - y_i)$$

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Extra trick. If the degree of f is sufficiently smaller than the degree of the bounding monomial, then we can decide to fix the positions of some elements of S to get a lower-degree polynomial to work with.

Additional trick - $\mathbb{Z}_p \times \mathbb{Z}_2$ example

Let $G = \mathbb{Z}_p \times \mathbb{Z}_2$, $S \subseteq G \setminus \{(0,0)\}$, $|S| = 7$ and type $(5,2)$, $\mathbf{a} = (0,0,1,0,0,0,1)$, $\mathbf{b} = (0,0,0,1,1,1,1,0)$.

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We desire a sequencing of the form

$$((x_1,0),(x_2,0),(x_3,1),(x_4,0),(x_5,0),(x_6,0),(x_7,1))$$

with partial sums

$$((y_0,0),(y_1,0),(y_2,0),(y_3,1),(y_4,1)(y_5,1),(y_6,1),(y_7,0))$$

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The polynomial is

$$\begin{aligned} f =&(x_2-x_1)(x_4-x_1)(x_5-x_1)(x_6-x_1)(x_4-x_2)(x_5-x_2)\\ &(x_6-x_2)(x_7-x_3)(x_5-x_4)(x_6-x_4)(x_6-x_5)(y_2-y_0)\\ &(y_7-y_1)(y_7-y_2)(y_5-y_3)(y_6-y_3)(y_6-y_4) \end{aligned}$$

Additional trick - $\mathbb{Z}_p \times \mathbb{Z}_2$ example

The polynomial f has degree 17. We need to find a monomial that divides $x_1^4 x_2^4 x_3 x_4^4 x_5^4 x_6^4 x_7$ of degree 22.

Fix $x_3 = c_1$ and $x_6 = c_2$ where $(c_1, 1), (c_2, 0) \in S$.

Hence we get the following simplified polynomial

$$\begin{aligned} f' = &(x_2 - x_1)(x_4 - x_1)(x_5 - x_1)(x_4 - x_2)(x_5 - x_2)(x_5 - x_4) \\ &(x_1 + x_2)(x_4 + x_5)(x_2 + c_1 + x_4 + x_5 + c_2 + x_7) \\ &(c_1 + x_4 + x_5 + c_2 + x_7)(x_4 + x_5 + c_2)(x_5 + c_2) \end{aligned}$$

In this case we need a monomial that divides $x_1^3 x_2^3 x_4^3 x_5^3$. The degree of f' is 12 and the monomial $x_1^3 x_2^3 x_4^3 x_5^3$ has coefficient -2 in f'. Therefore S has a sequencing.

Theorem (Costa, Della Fiore, Ollis, Rovner-Frydman)

Let $p > 5$ be prime and let $G = \mathbb{Z}_p \times \mathbb{Z}_2 \cong \mathbb{Z}_{2p}$, $S \subseteq G \setminus \{(0,0)\}$, $|S| = 10$. Then S is sequenceable.

There are 11 types to consider in this case

Type	**a**	deg	monomial/s	coefficient/s
$(10,0)$	$(0,0,0,0,0,0,0,0,0,0)$	89	$x_1^8x_2^9x_3^9x_4^9x_5^9x_6^9x_7^9x_8^9x_9^9x_{10}^9$ $x_1^9x_2^8x_3^9x_4^9x_5^9x_6^9x_7^9x_8^9x_9^9x_{10}^9$	$2^5 \cdot 7 \cdot 11^2 \cdot 21966239$ $2 \cdot 13 \cdot 211 \cdot 256046627$
$(9,1)$	$(0,0,0,0,0,1,0,0,0,0)$	52	$x_2^2x_3^4x_4^7x_5^8x_7^7x_8^8x_9^8x_{10}^8$	$-1 \cdot 2^2$
$(8,2)$	$(0,1,0,0,0,0,1,0,0,0)$	45	$x_1x_3x_4^7x_5^7x_6^7x_7x_8^7x_9^7x_{10}^7$	$-1 \cdot 2 \cdot 3 \cdot 7$
$(7,3)$	$(0,0,0,0,1,0,0,0,1,1)$	42	$x_2^6x_3^6x_4^6x_5^2x_6^6x_7^6x_8^6x_9^2x_{10}^2$	$-1 \cdot 2 \cdot 3 \cdot 7$
$(6,4)$	$(0,0,0,1,0,0,0,1,1,1)$	39	$x_1^5x_2^5x_3^5x_4^3x_5^5x_6^5x_7^3x_8^3x_9^3x_{10}^2$	$2 \cdot 5$
$(5,5)$	$(0,0,0,1,0,0,1,1,1,1)$ $(0,1,0,1,0,1,0,1,0,1)$	40 40	$x_1^4x_2^4x_3^4x_4^4x_5^4x_6^4x_7^4x_8^4x_9^4x_{10}^4$ $x_1^4x_2^4x_3^4x_4^4x_5^4x_6^4x_7^4x_8^4x_9^4x_{10}^4$	$2^2 \cdot 157$ $5 \cdot 19 \cdot 41 \cdot 83$

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Type	**a**	deg	monomial/s	coefficient/s
$(4,6)$	$(0,1,0,1,1,1,1,0,1,0)$	41	$x_1^2x_2^5x_3^3x_4^5x_5^5x_6^5x_7^5x_8^3x_9^5x_{10}^3$ $x_1^3x_2^4x_3^3x_4^5x_5^5x_6^5x_7^5x_8^3x_9^5x_{10}^3$	$2^4 \cdot 3 \cdot 5 \cdot 13$ $2 \cdot 3 \cdot 463$
$(3,7)$	$(0,0,1,0,1,1,1,1,1,1)$	46	$x_2^2x_3^6x_4^2x_5^6x_6^6x_7^6x_8^6x_9^6x_{10}^6$	$-1 \cdot 2^3 \cdot 3^2$
$(2,8)$	$(0,1,0,1,1,1,1,1,1,1)$	51	$x_1x_2x_3x_4^6x_5^7x_6^7x_7^7x_8^7x_9^7x_{10}^7$ $x_1x_3x_4^7x_5^7x_6^7x_7^7x_8^7x_9^7x_{10}^7$	$-1 \cdot 2 \cdot 1277$ $-1 \cdot 2 \cdot 17^2$
$(1,9)$	$(1,0,1,1,1,1,1,1,1,1)$	60	$x_1^2x_3^2x_4^8x_5^8x_6^8x_7^8x_8^8x_9^8x_{10}^8$ $x_1^2x_3^3x_4^7x_5^8x_6^8x_7^8x_8^8x_9^8x_{10}^8$	$2 \cdot 17^2$ $2^2 \cdot 647$
$(0,10)$	$(1,1,1,1,1,1,1,1,1,1)$	69	$x_1^2x_2^2x_3^4x_4^9x_5^7x_6^9x_7^9x_8^9x_9^9x_{10}^9$	$2^5 \cdot 3^2 \cdot 5$

Theorem (Costa, Della Fiore, Ollis, Rovner-Frydman)

Let $n = pt$ with p prime. Then subsets S of size k of $\mathbb{Z}_n \setminus \{0\}$ are sequenceable in the following cases:

- $k \leq 11$ *and* $t \leq 5$,
- $k = 12$ *and* $t \leq 4$,
- $k = 13$ *and* $t \in \{2, 3\}$, *provided* S *contains at least one element not in the subgroup of order* p,
- $k = 14$ *and* $t = 2$, *provided* S *contains at least one element not in the subgroup of order* p,
- $k = 15$ *and* $t = 2$, *provided* S *does not contain exactly* 0, 1, 2 *or* 15 *elements of the subgroup of order* p.

Using linear algebra arguments we have obtained the following asymptotic results.

Theorem (Costa, Della Fiore, Ollis, Rovner-Frydman)

Let $n = mt$ where all the prime factors of m are bigger than $k!/2$. Then subsets S of size k of $\mathbb{Z}_n \setminus \{0\}$ are sequenceable in the same cases of the previous theorem.

Thank you.